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# ROBIN'S INEQUALITY FOR 20-FREE INTEGERS

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## Abstract

In 1984, Robin showed that the Riemann Hypothesis for  $\zeta$  is equivalent to demonstrating  $\sigma(n) < e^\gamma n \log \log n$  for all  $n > 5040$ . Robin's inequality has since been proven for various infinite families of power-free integers: 5-free integers, 7-free integers, and 11-free integers. We extend these results to cover 20-free integers.

In 1984, Robin gave an equivalent statement of the Riemann Hypothesis for  $\zeta$  involving the divisors of integers.

**Theorem 1** (Robin [11]). *The Riemann Hypothesis is true if and only if for all  $n > 5040$ ,*

$$\sigma(n) < e^\gamma n \log \log n, \tag{RI}$$

where  $\sigma(n)$  is the sum of divisors function and  $\gamma$  is the Euler–Mascheroni constant.

Since then, (RI) has become known as Robin's inequality. There are twenty-six known counterexamples to (RI), of which 5040 is the largest [5].

Robin's inequality has been proven for various infinite families of integers, in particular the  $t$ -free integers. Recall that  $n$  is called  $t$ -free if  $n$  is not divisible by the  $t$ th power of any prime number, and  $t$ -full otherwise. In 2007, Choie, Lichiardopol, Moree, and Solé [4] showed that (RI) holds for all 5-free integers greater than 5040. Then, in 2012, Planat and Solé [12] improved this result to (RI) for 7-free integers greater than 5040, which was followed by Broughan and Trudgian [3] with (RI) for 11-free integers greater than 5040 in 2015. By updating Broughan and Trudgian's work, we prove our main theorem.

**Theorem 2.** *Robin's inequality holds for 20-free integers greater than 5040.*

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Since there are no 20-full integers less than 5041, we may give a cleaner statement for Robin's theorem.

**Corollary 1.** *The Riemann Hypothesis is true if and only if (RI) holds for all 20-full integers.*

## 1. A Bound for $t$ -free Integers

Solé and Planat [12] introduced the generalised Dedekind  $\Psi$  function

$$\Psi_t(n) := n \prod_{p|n} (1 + p^{-1} + \cdots + p^{-(t-1)}) = n \prod_{p|n} \frac{1 - p^{-t}}{1 - p^{-1}}.$$

Since

$$\sigma(n) = n \prod_{p^a || n} (1 + p^{-1} + \cdots + p^{-a}),$$

we see that  $\sigma(n) \leq \Psi_t(n)$ , provided that  $n$  is  $t$ -free. Thus, we study the function

$$R_t(n) := \frac{\Psi_t(n)}{n \log \log n}.$$

By Proposition 2 of [12], it is sufficient to consider  $R_t$  only at the primorial numbers  $p_n\# = \prod_{k=1}^n p_k$  where  $p_k$  is the  $k$ th prime. Compare this to the role of colossally abundant numbers in (RI) by Robin [11].

Using equation (2) of Broughan and Trudgian [3], we have for  $n \geq 2$

$$R_t(p_n\#) = \frac{p_n\# \prod_{p \leq p_n} \frac{1-p^{-t}}{1-p^{-1}}}{p_n\# \log \log p_n\#} = \frac{\prod_{p > p_n} (1-p^{-t})^{-1}}{\zeta(t) \log \vartheta(p_n)} \prod_{p \leq p_n} (1-p^{-1})^{-1}$$

where  $\vartheta(x)$  is the Chebyshev function  $\sum_{p \leq x} \log p$ .

In Sections 2 and 3, we construct two non-increasing functions,  $g_B(w; t)$  and  $g_\infty(w; t)$  such that for some constants  $x_0, B$  we have for  $x_0 \leq p_n \leq B$

$$g_B(p_n; t) \geq R_t(p_n\#) \exp(-\gamma)$$

and for  $p_n > B$

$$g_\infty(p_n; t) \geq R_t(p_n\#) \exp(-\gamma).$$

For a given  $t \geq 2$ , if we can show that all  $t$ -free numbers  $5040 < n \leq p_k\#$  satisfy (RI), that  $g_B(p_k; t) < 1$  and that  $g_\infty(B; t) < 1$ , then we are done.

## 2. Deriving $g_B(p_n; t)$

We start with some lemmas.

**Lemma 1.** *Let  $\rho$  be a non-trivial zero of the Riemann zeta function with positive imaginary part not exceeding  $3 \cdot 10^{12}$ . Then  $\Re \rho = 1/2$ .*

*Proof.* See Theorem 1 of [7]. □

**Lemma 2.** *Let  $B = 2.169 \cdot 10^{25}$ . Then we have*

$$|\vartheta(x) - x| \leq \frac{1}{8\pi} \sqrt{x} \log^2 x \quad \text{for } 599 \leq x \leq B.$$

*Proof.* Given that one knows the Riemann Hypothesis to height  $T$ , [1] tells us that we may use Schoenfeld's bounds from [10] but restricted to  $B$  such that

$$4.92 \sqrt{\frac{B}{\log B}} \leq T.$$

Using  $T = 3 \cdot 10^{12}$  from Lemma 1 we find  $B = 2.169 \cdot 10^{25}$  is admissible. □

**Lemma 3.** *Let  $\log x \geq 55$ . Then*

$$|\vartheta(x) - x| \leq 1.388 \cdot 10^{-10} x + 1.4262 \sqrt{x}$$

or

$$|\vartheta(x) - x| \leq 1.405 \cdot 10^{-10} x.$$

*Proof.* From Table 1 of [6] we have for  $x > \exp(55)$

$$|\psi(x) - x| \leq 1.388 \cdot 10^{-10} x$$

so that by Theorem 13 of [9] we get, again for  $x > \exp(55)$ , that

$$|\vartheta(x) - x| \leq 1.388 \cdot 10^{-10} x + 1.4262 \sqrt{x}.$$

The second bound follows trivially. □

**Lemma 4.** *Take  $B$  as above and define*

$$C_1 = \int_B^\infty \frac{(\vartheta(t) - t)(1 + \log t)}{t^2 \log^2 t} dt.$$

*Then  $C_1 \leq 2.645 \cdot 10^{-9}$ .*

*Proof.* We split the integral at  $X_0 = \exp(2000)$ , apply Lemma 3 and consider

$$1.405 \cdot 10^{-10} \int_B^{X_0} \frac{1 + \log t}{t \log^2 t} dt \leq 1.430 \cdot 10^{-10} \int_B^{X_0} \frac{dt}{t \log t} \leq 5.055 \cdot 10^{-10}.$$

For the tail of the integral, we use

$$|\vartheta(x) - x| \leq 30.3x \log^{1.52} x \exp(-0.8\sqrt{\log x})$$

from Corollary 1 of [8], valid for  $x \geq X_0$ . We can then majorise the tail with

$$30.3 \int_{X_0}^{\infty} \frac{\log t \exp(-0.8\sqrt{\log t})}{t} dt$$

which is less than  $2.139 \cdot 10^{-9}$ .

□

**Lemma 5.** Take  $B, C_1$  as above and let  $599 \leq x \leq B$ . For  $t > 1$ , define

$$w(t) = \frac{(\log t + 3)\sqrt{B} - (\log B + 3)\sqrt{t}}{4\pi\sqrt{tB}}.$$

Then

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \geq \frac{\exp(-\gamma)}{\log x} \exp\left(\frac{1.02}{(x-1)\log x} + \frac{\log x}{8\pi\sqrt{x}} + C_1 + w(x)\right).$$

*Proof.* Let  $M$  be the Meissel-Mertens constant

$$M = \gamma + \sum_p (\log(1 - 1/p) + 1/p).$$

Then by 4.20 of [9] we have

$$\left| \sum_{p \leq x} \frac{1}{p} - \log \log x - M \right| \leq \frac{|\vartheta(x) - x|}{x \log x} + \int_x^{\infty} \frac{|\vartheta(t) - t|(1 + \log t)}{t^2 \log^2 t} dt.$$

Since  $599 \leq x \leq B$  we can use Lemma 2 to bound the first term with

$$\frac{\log x}{8\pi\sqrt{x}}.$$

We can split the integral at  $B$  and over the range  $[B, \infty)$  use the bound from Lemma 4. This leaves the range  $[x, B]$  where we can use Lemma 2 and a straightforward integration yields a contribution of

$$\frac{(\log x + 3)\sqrt{B} - (\log B + 3)\sqrt{x}}{4\pi\sqrt{xB}} = w(x).$$

We then simply follow the method used to prove Theorem 5.9 of [6] with our bounds in place of

$$\frac{\eta_k}{k \log^k x} + \frac{(k+2)\eta_k}{(k+1) \log^{k+1} x}.$$

□

We also need Lemma 2 of [12].

**Lemma 6** (Solé and Planat [12]). *For  $n \geq 2$ ,*

$$\prod_{p > p_n} \frac{1}{1 - p^{-t}} \leq \exp(2/p_n).$$

Putting all this together, we have the following.

**Lemma 7.** *Let  $w(t)$  be as per Lemma 5. Now define*

$$g_B(p_n; t) = \frac{\exp\left(\frac{2}{p_n} + \frac{1.02}{(p_n-1) \log p_n} + \frac{\log p_n}{8\pi\sqrt{p_n}} + C_1 + w(p_n)\right) \log p_n}{\zeta(t) \log\left(p_n - \frac{\sqrt{p_n} \log^2 p_n}{8\pi}\right)}.$$

*Then for  $t \geq 2$  and  $599 \leq p_n \leq B = 2.169 \cdot 10^{25}$  we have  $g_B(p_n; t)$  non-increasing in  $n$  and  $R_t(p_n\#) \leq \exp(\gamma)g_B(p_n; t)$ .*

### 3. Deriving $g_\infty(p_n; t)$

We will need a further bound.

**Theorem 3.** *For  $x \geq 767\,135\,587$ ,*

$$\prod_{p \leq x} \frac{p}{p-1} \leq e^\gamma \log x \exp\left(\frac{1.02}{(x-1) \log x} + \frac{1}{6 \log^3 x} + \frac{5}{8 \log^4 x}\right).$$

*Proof.* This is the last display on page 245 of [6] with  $k = 3$  so that  $\eta_k = 0.5$ . □

We can now deduce

**Theorem 4.** *Define*

$$g_\infty(p_n; t) = \frac{\exp\left(\frac{2}{p_n} + \frac{1.02}{(p_n-1) \log p_n} + \frac{1}{6 \log^3 p_n} + \frac{5}{8 \log^4 p_n}\right) \log p_n}{\zeta(t) \log(p_n - 1.338 \cdot 10^{-10} p_n - 1.4262\sqrt{p_n})}.$$

*Then for  $t \geq 2$  and  $\log p_n \geq 55$  we have*

$$R_t(p_n\#) \leq e^\gamma g_\infty(p_n; t)$$

*and  $g_\infty(p_n; t)$  is non-increasing in  $n$ .*

#### 4. Computations

The proof rests on Briggs' work [2] on the colossally abundant numbers, which implies (RI) for  $5040 < n \leq 10^{(10^{10})}$ . We extend this result with the following theorem.

**Theorem 5.** *Robin's inequality holds for all  $5040 < n \leq 10^{(10^{13.11485})}$ .*

*Proof.* We implemented Brigg's algorithm from [2] but using extended precision (100 bits) and interval arithmetic to carefully manage rounding errors. The final  $n$  checked was

$$29\,996\,208\,012\,611\# \cdot 7\,662\,961\# \cdot 44\,293\# \cdot 3\,271\# \cdot 666\# \cdot 233\# \cdot 109\# \cdot 61\# \\ \cdot 37\# \cdot 23\# \cdot 19\# \cdot (13\#)^2 \cdot (7\#)^4 \cdot (5\#)^3 \cdot (3\#)^{10} \cdot 2^{19}.$$

□

**Corollary 2.** *Robin's inequality holds for all  $13\# \leq n \leq 29\,996\,208\,012\,611\#$ .*

We are now in a position to prove Theorem 2. We find that

$$g_B(29\,996\,208\,012\,611; 20) < 1$$

and

$$g_\infty(B; 20) < 1$$

and the result follows.

#### 5. Comments

In terms of going further with this method, we observe that both

$$g_B(29\,996\,208\,012\,611; 21) > 1$$

and

$$g_\infty(B; 21) > 1$$

so one would need improvements in both. We only pause to note that one of the inputs to Dusart's unconditional bounds that feed into  $g_\infty$  is again the height to which the Riemann Hypothesis is known<sup>3</sup>, so the improvements from Lemma 1 could be incorporated.

Finally, we observe that if  $R_t(p_n\#)$  could be shown to be decreasing in  $n$ , then our lives would have been much easier.

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<sup>3</sup>Dusart uses  $T \geq 2\,445\,999\,556\,030$ .

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